ON THE APPROXIMATE SOLUTION OF A CERTAIN TYPE OF INTEGRAL EQUATION

(O PRIBLIZHENNOM RESHENII ODNOGO TIPA INTEGRAL'NYKH URAVNENII)

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V.M. ALEKSANDROV
(Rostov-on-Don)

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Many contact problems of the theory of elasticity (for example, contact problems for an elastic layer, and others) can be reduced to integral equations of the type

$$\int_{-1}^{1} \varphi(\xi) K\left(\frac{x-\xi}{\lambda}\right) d\xi = \pi f(x), \qquad (|x| \leqslant 1, \ \lambda \in (0, \infty)) \tag{0.1}$$

The parameter λ characterizes the thickness of the layer, and so on (the properties of the kernel of the integral equation are given below). In the sequel, we consider several methods of solutions of this equation, each of which is suited for use on some interval of variation of λ; combined, these methods cover the total interval over which λ can vary. Namely, for large and for very small values of λ the solutions of Equation (0.1) are obtained in the form of simple sufficiently accurate formulas; for intermediate values of λ, the solution of Equation (0.1) is reduced to the solution of an easily constructed system of linear algebraic equations.

In the nature of illustrative examples of our methods, we consider the effect of a stamp (die) on an elastic layer resting without friction on a solid base, and also on a layer which is rigidly connected with a nondeformable base. We note that the first one of these problems was considered in [1-7]. In contrast with those works, the methods presented here will make it possible to find, in a considerably simpler way, practical exact solutions of the mentioned problems for arbitrary values of λ. The bounds of validity of each method are indicated, and necessary tables are given for the use of our methods.

1. General form of the solution of the integral equation, degenerate solution. We shall consider an integral equation of the type (0.1) and shall assume that its kernel has the following properties.

1. For every $k \in (-\infty, \infty)$

$$K(k) = -\ln|k| + F(k) \qquad (F(k) \in H_q^\alpha\ (-c, c),\ \alpha > 0,\ q \geqslant 1,\ c < \infty) \tag{1.1}$$

where $F(k)$ is an even function which can be represented as a power series with a radius of convergence $\rho (0 \leqslant \rho \leqslant \infty)$

$$F(k) = \sum_{i=0}^{\infty} a_i k^{2i} \qquad (|k| < \rho) \tag{1.2}$$

2. When $|k| \to \infty$, the kernel satisfies the relation

$$K(k) \to \pi A \delta(k) \qquad (A = \text{const}) \tag{1.3}$$

where $\delta(k)$ is Dirac's delta-function.

In what follows, we shall assume also that

$$f(x) \in H_p{}^\alpha (-1,1), \quad \alpha > 0, \quad p \geqslant 1$$

where $H_n{}^\alpha(-\beta, \beta)$ denotes the space of functions whose nth derivatives satisfy Hoelder's condition with index α when $-\beta \leqslant x \leqslant \beta$.

Let us determine the general form of the solution of the integral equation. Making use of Formula (1.1), we may express Equation (0.1) in the form

$$-\int_{-1}^{1} \varphi(\xi) \ln \frac{|x-\xi|}{\lambda} d\xi = \pi f(x) - \int_{-1}^{1} \varphi(\xi) F\left(\frac{x-\xi}{\lambda}\right) d\xi, \qquad |x| \leqslant 1 \tag{1.4}$$

We shall look for a solution $\varphi(\xi)$ of Equation (1.4) in the class $L(-1, 1)$ of functions which have summable absolute values. In this case

$$\psi(x) = f(x) - \frac{1}{\pi} \int_{-1}^{1} \varphi(\xi) F\left(\frac{x-\xi}{\lambda}\right) d\xi \in H_r{}^\alpha (-1,1) \qquad (\alpha > 0,\ r = \operatorname{Inf}(p, q)) \tag{1.5}$$

If the function $\psi(x)$ is assumed to be known for the time being, then Equation (1.4) will take on the form

$$-\int_{-1}^{1} \varphi(\xi) \ln \frac{|x-\xi|}{\lambda} d\xi = \pi \psi(x), \qquad |x| \leqslant 1 \tag{1.6}$$

Its solution in the class $L(-1, 1)$ has the form [8]

$$\varphi(x)=\frac{1}{\pi\sqrt{1-x^2}}\left[P-\int_{-1}^{1}\frac{\psi'(t)\sqrt{1-t^2}}{t-x}\,dt\right] \tag{1.7}$$

$$P=\int_{-1}^{1}\varphi(\xi)\,d\xi=\frac{1}{\ln 2\lambda}\int_{-1}^{1}\frac{\psi(t)\,dt}{\sqrt{1-t^2}} \tag{1.8}$$

Substituting $\psi(x)$ from (1.5) into Formulas (1.7), and (1.8), we obtain

$$\varphi(x)=\frac{1}{\pi\sqrt{1-x^2}}\left[P-\int_{-1}^{1}\frac{f'(t)\sqrt{1-t^2}}{t-x}\,dt+\right.$$

$$\left.+\frac{1}{\pi\lambda}\int_{-1}^{1}\varphi(\xi)\,d\xi\int_{-1}^{1}F'\left(\frac{t-\xi}{\lambda}\right)\frac{\sqrt{1-t^2}\,dt}{t-x}\right] \tag{1.9}$$

$$P=\frac{1}{\ln 2\lambda}\left[\int_{-1}^{1}\frac{f(t)\,dt}{\sqrt{1-t^2}}-\frac{1}{\pi}\int_{-1}^{1}\varphi(\xi)\,d\xi\int_{-1}^{1}F\left(\frac{t-\xi}{\lambda}\right)\frac{dt}{\sqrt{1-t^2}}\right] \tag{1.10}$$

We have thus reduced the integral Equation (0.1) of the first kind to an integral equation of the second kind (1.9) under the condition (1.10).

For what follows, we need to establish some properties of the integral

$$I(x)=\int_{-1}^{1}\frac{\gamma(t)\sqrt{1-t^2}}{t-x}\,dt \tag{1.11}$$

that are related to the properties of the function $\gamma(t)$. The following theorem is valid.

Theorem. Let

$$\gamma(t)\in H_m^{\alpha}(-1,1) \qquad (\alpha>0)$$

Then the function

$$I(x)\in C_m(-1,1)$$

On the basis of this formula and from Formulas (1.5), (1.7) and (1.8) one can easily conclude that a solution of Equation (0.1) in the class $L(-1,1)$ must, for every $\lambda\in(0,\infty)$, have the form

$$\varphi(x)=\frac{\Phi(x)}{\sqrt{1-x^2}}, \qquad (\Phi(x)\in C_{r-1}(-1,1),\ r=\inf(p,q)) \tag{1.12}$$

If in (1.5) the value of $\alpha > 1/2$, then one can show, in the same way, that the solutions of (0.1) which are bounded for $x=1$ ($x=-1$) must have the form

$$\varphi(x) = \sqrt{\frac{1-x}{1+x}}\,\Phi_1(x), \qquad (\Phi_1(x) \in C_{r-1}(-1,1)) \tag{1.13}$$

(analogously, if the solution is bounded at $x = -1$)

$$\varphi(x) = \sqrt{1-x^2}\,\Phi_2(x) \qquad (\Phi_2(x) \in C_{r-1}(-1,1)) \tag{1.14}$$

Here, we impose on $f(x)$ either one, or two bounds, respectively.

Let us now find the degenerate (limiting) solution of Equation (0.1) for either very large or very small values of λ. For large values of the parameter λ, one may set $F([t-\xi]/\lambda) = F(0)$, $F'([t-\xi]/\lambda) = 0$. From Formulas (1.9) and (1.10) we now obtain

$$\varphi(x) = \frac{1}{\pi\sqrt{1-x^2}}\left[P - \int_{-1}^{1} \frac{f'(t)\sqrt{1-t^2}}{t-x}\,dt\right], \quad P = \frac{1}{\ln 2\lambda + F(0)} \int_{-1}^{1} \frac{f(t)\,dt}{\sqrt{1-t^2}} \tag{1.15}$$

The bounded solutions, and the corresponding conditions imposed on the function $f(x)$, can be obtained quite easily [8] from the first formula of (1.15). For very small values of the parameter λ Equation (0.1) can be expressed, in accordance with Formula (1.3), in the form

$$A \int_{-1}^{1} \varphi(\xi)\,\delta\left(\frac{x-\xi}{\lambda}\right) d\xi = f(x) \qquad (|x| \leqslant 1) \tag{1.16}$$

The solution of Equation (1.16) can be found without difficulty*

$$\varphi(x) = \frac{f(x)}{A\lambda} \qquad (|x| \leqslant 1) \tag{1.17}$$

In case λ is small, the bounded solutions correspond to the solutions which vanish when $x = 1$ ($x = -1$) and $x = \pm 1$. The conditions imposed on $f(x)$ in this case are, obviously, of the form

$$f(1) = 0 \qquad (f(-1) = 0), \qquad f(\pm 1) = 0 \tag{1.18}$$

A comparison of (1.17) and (1.12) reveals that the solution (1.17)

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* A more exact solution of Equation (0.1) for small values of λ can be obtained by the method of successive approximations [9], whereby the zeroth approximation will be the solution (1.17). However, in practice this method leads to the difficulties of numerical integration at each stage of the successive approximations, and to the accumulation of errors.

will yield, obviously, true results when $|x| \leqslant 1 - \varepsilon(\lambda)$ where $\varepsilon(\lambda) > 0$ and tends to zero as $\lambda \to 0$.

In conclusion of this section, we note the important case when $f(x) = \mu + \nu x$ in Equation (0.1). The essential point here is, that it is, frequently, sufficient to determine only the quantities

$$P = \int_{-1}^{1} \varphi(\xi)\, d\xi, \qquad M = \int_{-1}^{1} \xi\varphi(\xi)\, d\xi \tag{1.19}$$

in place of the solution of Equation (0.1) itself.

These quantities can be determined for an arbitrary function $f(x)$ if the solution (or its approximation) of Equation (0.1), $\varphi(x) = \mu\varphi_\mu(x) + \nu\varphi_\nu(x)$, for the function $f(x) = \mu + \nu x$ is known. Indeed, making use of a method of Mossakovskii [10] we can obtain

$$P = \int_{-1}^{1} f(x)\, \varphi_\mu(x)\, dx, \qquad M = \int_{-1}^{1} f(x)\, \varphi_\nu(x)\, dx \tag{1.20}$$

2. Solution for large values of the parameter λ. Making use of Formula (1.12), we express Equation (1.9) in the form

$$\Phi(x) = \frac{1}{\pi}\left[P - \int_{-1}^{1} \frac{f'(t)\sqrt{1-t^2}}{t-x}\, dt\right] + \frac{1}{U^2\lambda} \int_{-1}^{1} \Phi(\xi)\, d\xi \int_{-1}^{1} F'\left(\frac{t-\xi}{\lambda}\right) \frac{\sqrt{1-t^2}\, dt}{(t-x)\sqrt{1-\xi^2}} \tag{2.1}$$

Considering the quantity P as having been given, we can determine $\Phi(x)$ by means of Formula (2.1), and after that find $\varphi(x)$ and P with the aid of Formulas (1.12) and (1.10).

Treating the right-hand side of Equation (2.1) as an operator which maps the elements of the space $C(-1, 1)$ into elements of the same space, and making use of Banach's "fixed point principle" we can show that the unique solution of Equation (2.1) can be obtained by the method of successive approximations for sufficiently large values of the parameter λ, namely, when

$$\lambda > \frac{\max|F''|}{-\max|F'| + \sqrt{(\max|F'|)^2 + 2\max|F''|}} \tag{2.2}$$

At the same time, for sufficiently large values of λ (when $\lambda > 2/\rho$), we can rewrite (2.1) on the basis of Formula (1.2) in the more useful form

$$\Phi(x) = \Phi^\circ(x) + \sum_{i=1}^{\infty} \frac{1}{\lambda^{2i}} \int_{-1}^{1} \Phi(\xi) L_i(\xi, x)\, d\xi \tag{2.3}$$

$$\Phi^\circ(x) = \frac{1}{\pi}\left[P - \int_{-1}^{1} \frac{f'(t)\sqrt{1-t^2}}{t-x}\, dt\right]$$

$$L_i(\xi, x) = \frac{2ia}{\pi^2} \int_{-1}^{1} \frac{(t-\xi)^{2i-1}\sqrt{1-t^2}}{(t-x)\sqrt{1-\xi^2}}\, dt \tag{2.4}$$

The solution of Equation (2.3), obviously, can be found by the method of successive approximations when

$$\lambda > \lambda^*, \qquad \lambda^* = \sup\left\{\frac{2}{\rho}, \frac{\max|F''|}{-\max|F'| + \sqrt{(\max|F|)^2 + 2\max|F''|}}\right\} \tag{2.5}$$

For the initial approximation it is convenient to take $\Phi^\circ(x)$; it is easy to see that the nth approximation will have the form

$$\Phi^n(x) = \sum_{k=0}^{\infty} \Phi_{2k}^n(x) \frac{1}{\lambda^{2k}} \tag{2.6}$$

The solution of Equation (2.3) can be obtained also in a different way, which is as follows. We look for a solution in the form

$$\Phi(x) = \sum_{k=0}^{\infty} \Phi_{2k}(x) \frac{1}{\lambda^{2k}} \tag{2.7}$$

Substituting Expression (2.7) into Equation (2.3), and equating the coefficients of like powers of λ^{-2}, we obtain

$$\Phi_0(x) = \Phi^\circ(x), \qquad \Phi_2(x) = \int_{-1}^{1} \Phi_0(\xi) L_1(\xi, x)\, d\xi$$

$$\Phi_4(x) = \int_{-1}^{1} [\Phi_0(\xi) L_2(\xi, x) + \Phi_2(\xi) L_1(\xi, x)]\, d\xi \tag{2.8}$$

Determining successively $\Phi_0(x)$, $\Phi_2(x)$, and $\Phi_4(x)$ we obtain the solution of (2.3) with an accuracy up to the term of order λ^{-6}. After this we obtain with the aid of Formula (1.12) the function

$$\varphi(x) = \frac{P}{\pi\sqrt{1-x^2}}\left[1 + \frac{2a_1}{\lambda^2}\left(\frac{1}{2} - x^2\right) + \frac{4a_2}{\lambda^4}\left(\frac{7}{8} - x^2 - x^4\right)\right] -$$

$$- \frac{1}{\pi\sqrt{1-x^2}} \int_{-1}^{1} \sqrt{1-t^2} f'(t) \left\{\frac{1}{t-x} - \frac{2a_1 x}{\lambda^2} - \frac{1}{\lambda^4} \times\right.$$

$$\times\,[2a_2(6x^3-6x^2t+2xt^2-2x+3t)+2a_1^2x]\Big\}\,dt+O\left(\frac{1}{\lambda^4}\right) \tag{2.9}$$

Now, on the basis of (1.10) and with the aid of (1.2) and (2.9), we determine the quantity

$$P=\left[\ln 2\lambda+a_0+\frac{a_1}{\lambda^2}-\frac{a_1^2}{4\lambda^4}+\frac{9a_2}{4\lambda^4}+O\left(\frac{1}{\lambda^4}\right)\right]^{-1}\left[\int_{-1}^{1}\frac{f(t)\,dt}{\sqrt{1-t^2}}-\right.$$
$$\left.-\frac{1}{\lambda^2}\int_{-1}^{1}\sqrt{1-t^2}\,f'(t)\,t\left(a_1+\frac{a_2t^2}{\lambda^2}+\frac{7a_2}{2\lambda^2}\right)dt\right]+O\left(\frac{1}{\lambda^4}\right) \tag{2.10}$$

The bounded solutions and the corresponding conditions imposed on the function $f(x)$, can easily be obtained [5] from Formula (2.9).

The second method of solving Equation (2.3) can, obviously, always be used when (1.2) is valid, i.e. when $\lambda > 2/\rho$. The connection between the first and second method of solution consists of the following:

$$\Phi_{2k}^{n}(x)\to\Phi_{2k}(x)\quad \text{when } n\to\infty,\ \lambda>\lambda^* \tag{2.11}$$

because of the uniqueness of the solution under the specified values of the parameter λ, and the uniqueness of the representation of $\Phi(x)$ in the form (2.7).

It is not difficult to show that for any n, the series (2.6) is absolutely convergent when $\lambda > 2/\rho$. It is obvious that the series (2.7), which is a solution of (2.3), will then converge absolutely for $\lambda > \lambda^*$.

3. Approximate solution for arbitrary values of the parameter λ. Let the parameter λ be kept fixed, and let the function $F(k)$ be approximated in $0 \leqslant |k| \leqslant 2/\lambda$ by means of a polynomial in some way (for example, by interpolation)

$$F(k)=a_0+\sum_{r=1}^{i}b_rk^{2r} \tag{3.1}$$

Substituting Expression (3.1) into (1.4), we derive the following equation

$$-\int_{-1}^{1}\varphi(\xi)\ln\frac{|x-\xi|}{\lambda}\,d\xi=\pi f(x)-\sum_{r=0}^{i}\frac{b_r}{\lambda^{2r}}\int_{-1}^{1}\varphi(\xi)(x-\xi)^{2r}\,d\xi \tag{3.2}$$

We shall look for a solution of Equation (3.2) in the form

$$\varphi(\xi) = \varphi_0(\xi) + \varphi_1(\xi), \qquad \varphi_1(\xi) = \frac{1}{\sqrt{1-\xi^2}}\left(\sum_{s=-1}^{i-1} A_{2s+2}\,\xi^{2s+2} + \sum_{s=0}^{i-1} A_{2s+1}\,\xi^{2s+1}\right) \quad (3.3)$$

The function $\varphi_0(\xi)$ is determined by means of the Formulas (1.15); the constants A_{2s+2} and A_{2s+1} are to be determined.

Substituting $\varphi(\xi)$ from (3.3) into (3.2), and evaluating the integrals, we obtain the following relations:

$$\sum_{s=-1}^{i-1} A_{2s+2}\left(\frac{(2s+1)!!}{(2s+2)!!}[\ln 2\lambda + a_0 - \gamma(2s+3)] + P_{2s+2}(x) + \sum_{r=1}^{i} \frac{b_r}{\lambda^{2r}} P_{2r}^{2s+2}(x)\right) =$$

$$= -\sum_{r=1}^{i} \frac{b_r}{\lambda^{2r}} P^{2r}(x) \qquad (3.4)$$

$$\sum_{s=0}^{i-1} A_{2s+1}\left(Q_{2s+1}(x) - \sum_{r=1}^{i} \frac{b_r}{\lambda^{2r}} Q_{2r}^{2s+1}(x)\right) = \sum_{r=1}^{i} \frac{b_r}{\lambda^{2r}} Q^{2r}(x) \qquad (3.5)$$

Here

$$P_{2s+2}(x) = \sum_{n=0}^{s} \frac{(2n-1)!!}{2n!!\,(2s-2n+2)}\, x^{2s-2n+2}$$

$$Q_{2s+1}(x) = \sum_{n=0}^{s} \frac{(2n-1)!!}{2n!!\,(2s-2n+1)}\, x^{2s-2n+1} \qquad (3.6)$$

$$P_{2r}^{2s+2}(x) = \sum_{m=-1}^{r-1} C_{2r}^{2m+2} \frac{(2m+2s+3)!!}{(2m+2s+4)!!}\, x^{2r-2m-2}$$

$$Q_{2r}^{2s+1}(x) = \sum_{m=0}^{r-1} C_{2r}^{2m+1} \frac{(2m+2s+1)!!}{(2m+2s+2)!!}\, x^{2r-2m-1}$$

$$P^{2r}(x) = \sum_{m=-1}^{r-1} C_{2r}^{2m+2} P_{2m+2}\, x^{2r-2m-2}, \qquad Q^{2r}(x) = \sum_{m=0}^{r-1} C_{2r}^{2m+1} M_{2m+1}\, x^{2r-2m-1}$$

$$\gamma(1) = 0, \qquad \gamma(2s+3) = \sum_{p=1}^{2s+2} \frac{(-1)^{p+1}}{p}, \qquad P_0 = \frac{1}{\pi(\ln 2\lambda + a_0)} \int_{-1}^{1} \frac{f(t)\,dt}{\sqrt{1-t^2}}$$

$$P_{2m+2} = \frac{1}{\pi(\ln 2\lambda + \alpha_0)} \frac{(2m+1)!!}{(2m+2)!!} \int_{-1}^{1} \frac{f(t)\,dt}{\sqrt{1-t^2}} +$$

$$+\frac{1}{\pi}\sum_{q=0}^{m}\frac{(2q-1)!!}{2q!!}\int_{-1}^{1}t^{2m-2q+1}\sqrt{1-t^2}\,f'(t)\,dt$$

$$M_{2m+1}=\frac{1}{\pi}\sum_{q=0}^{m}\frac{(2q-1)!!}{2q!!}\int_{-1}^{1}t^{2m-2q}\sqrt{1-t^2}\,f'(t)\,dt$$

Equating the coefficients of equal powers of x in the relation (3.4), we obtain $i+1$ equations for the determination of the $i+1$ quantities A_{2s+2}. In an analogous manner we obtain from the relation (3.5) i equations for the determination of the i quantities A_{2s+1}. Having solved the indicated systems of linear algebraic equations, we obtain the approximate solution of Equation (1.4) by means of Formulas (3.3) and (1.15). After the general solution has been found, it is easy to find the bounded solutions and the corresponding conditions imposed on the function $f(x)$.

It is obvious that for the finding of the approximate solution of equal accuracy, it is necessary to increase i with any decrease of λ. Hence, for very small λ it is advisable to use the degenerate solution (1.17). The convergence of the obtained approximate solutions, when $i\to\infty$, to the exact solution follows directly from the estimate (2.10) which was given in the work [2].

We note, also, that if $F(k)$ is continuous and differentiable any number of times in the interval $-\infty < k < \infty$, then the approximate solutions retain all the properties of the exact solution.

4. Examples. Let us consider the problem of the effect of a rigid* stamp [die] on an elastic layer (a) lying frictionless on a solid base, (b) rigidly attached to a solid base. There exist no frictional forces between the stamp and layer.

By the methods of the operational calculus, the problems (a) and (b) can be reduced to the solution of integral equations, which in dimensionless coordinates have the form (0.1); the function $\varphi(\xi)$ is now the unknown pressure between the stamp and the layer along the line of contact

$$\lambda=\frac{h}{a},\qquad f(x)=\frac{\Delta}{a}\,\delta(x)$$

$$\Delta=\frac{E}{2(1-\sigma^2)},\qquad x=\frac{x'}{a},\ \xi=\frac{\xi'}{a}$$

* The solution for the problem of an elastic stamp can also be obtained quite easily under the usual hypotheses that apply in that case.

Here, h is the thickness of the layer; a is the half-length of the line of contact; $\delta(x)$ is the depression of the boundary of the layer under the stamp, x' and ξ' are dimensional variables.

The kernels of the integral equations can be expressed in the following form:

$$\text{(a)} \quad K(k) = \int_0^\infty \frac{\cosh 2u - 1}{u\,(\sinh 2u + 2u)} \cos ku\, du \tag{4.1}$$

$$\text{(b)} \quad K(k) = \int_0^\infty \frac{[2\,(3 - 4\sigma)\sinh 2u - 4u]}{u\,[2\,(3 - 4\sigma)\cosh 2u + (3 - 4\sigma)^2 + 1 + 4u^2]} \cos ku\, du \tag{4.2}$$

It can be shown that the kernels (4.1) and (4.2) satisfy all the requirements mentioned in Section 1; the functions $F(k)$ which correspond to these kernels will be continuous and differentiable any number of times when $-\infty < k < \infty$, while the radius of convergence $\rho = 2$. It is easy to show that $\max|F''(k)| = 2a_1$, $\max|F'(k)| = 2B$.

TABLE 1.

	σ	a_0	a_1	a_2	a_3	B	A	λ_∞	λ°	λ^*	λ_0
(a)		—0.352	0.521	—0.135	0.0346	0.584	0.500	5.5	2.4	1.5	1/5
	0.1	—0.396	0.603	—0.190	0.0592	0.621	0.494	6	2.5	1.6	1/5
	0.2	—0.442	0.647	—0.212	0.0676	0.646	0.469	6	2.6	1.7	1/4.5
(b)	0.3	—0.527	0.716	—0.245	0.0801	0.688	0.408	6.5	2.8	1.8	1/7
	0.4	—0.683	0.828	—0.298	0.0998.	0.760	0.278	7.5	3.2	1.9	1/10

The values of the quantity B and of all the other quantities needed for the derivation of the degenerate solutions (1.15), (1.17), and of the solutions (2.9) and (2.10) are given in Table 1. The degenerate solutions for very large λ should be used when $\lambda > \lambda_\infty$; the degenerate solutions for very small λ should be used when $\lambda < \lambda_0$; the solutions for large λ should be used when $\lambda > \lambda^\circ$. Within these boundaries, the obtained approximate solutions can be considered as being exact for all practical purposes.

5. Examples (continuation). The approximate solutions of the problems (a) and (b) for $\lambda^\circ > \lambda > \lambda_0$ can be obtained by the method presented in Section 3. The values of $F(k)$ required for the application of this method have been computed on the computing machine "Ural" and are given in

Tables 2 and 3 for the problems (a) and (b), respectively. The tables are constructed in such a manner that the intermediate values of the function $F(k)$ can be obtained by linear interpolation, correct to three decimal places.

TABLE 2.

k	$F(k)$	k	$F(k)$	k	$F(k)$	k	$F(k)$	k	$F(k)$
0.00	−0.352	0.75	−0.096	1.50	0.389	2.25	0.798	3.00	1.096
0.05	−0.350	0.80	−0.066	1.55	0.420	2.30	0.821	3.05	1.113
0.10	−0.346	0.85	−0.035	1.60	0.451	2.35	0.844	3.10	1.129
0.15	−0.340	0.90	−0.003	1.65	0.481	2.40	0.866	3.15	1.146
0.20	−0.331	0.95	0.029	1.70	0.510	2.45	0.887	3.20	1.162
0.25	−0.320	1.00	0.062	1.75	0.540	2.50	0.908	3.25	1.177
0.30	−0.306	1.05	0.095	1.80	0.568	2.55	0.929	3.30	1.193
0.35	−0.290	1.10	0.128	1.85	0.596	2.60	0.949	3.35	1.208
0.40	−0.272	1.15	0.162	1.90	0.623	2.65	0.968	3.40	1.223
0.45	−0.252	1.20	0.195	1.95	0.650	2.70	0.988	3.45	1.238
0.50	−0.229	1.25	0.228	2.00	0.676	2.75	1.007	3.50	1.252
0.55	−0.206	1.30	0.261	2.05	0.702	2.80	1.025		From
0.60	−0.180	1.35	0.293	2.10	0.727	2.85	1.043		here
0.65	−0.153	1.40	0.326	2.15	0.751	2.90	1.061		on as
0.70	−0.125	1.45	0.358	2.20	0.775	2.95	1.079		in k

The computations show that in order to obtain sufficiently exact solutions for practical purposes, one must select the quantity i in Formula (3.1), in case of equal subdivisions of interpolation, in accordance with the relation

$$i \geqslant E^{\circ}\left(1+\frac{2}{\lambda}\right) \tag{5.1}$$

The function $E^{\circ}(x)$ is equal to the integer nearest to x. Let us consider, for example, the solution of the problems (a) and (b) for the case when $\lambda = 1$, $\delta(x) = \delta + \alpha(x)$ where we choose $i = 3$ in accordance with (5.1) (in the given example for the finding of the approximation (3.1) use was made of the roots of the Chebychev polynomials $T_7(k/2)$ for the points of interpolation on the interval $[0, 2]$; a choice of equal subdivisions would have given a less exact result).

$$\text{(a)}\quad \varphi(x) = \frac{\Delta}{\sqrt{1-x^2}}\left[\frac{\delta}{a}(1.913 - 0.979x^2 + 0.272x^4 - 0.082x^6) + \right.$$
$$\left. + \alpha x(1.788 - 0.649x^2 + 0.194x^4)\right] \tag{5.2}$$

$$(б)\quad \varphi(x)=\frac{\Delta}{\sqrt{1-x^2}}\left[\frac{\delta}{a}(2.334-1.400x^2+0.477x^4-0.172x^6)+\right.$$
$$\left.+\,\alpha x(2.163-1.123x^2+0.380x^4)\right]\quad(\sigma=0.3) \qquad (5.3)$$

One can easily prove that the solution (5.2) agrees quite well with an analogous solution obtained in the works [1-4,6].

Computations made by the author showed that in the case of problem (b) for equal depressions $\delta(x)$ and for arbitrary λ, the pressure $\varphi(x)$ increases with an increase of σ. Furthermore, it was shown that the pressure under the stamp, in case of problem (b), is greater than in the problem (a) under otherwise equal conditions and for arbitrary λ. This

TABLE 3.

k	$F(k)$	k	$F(k)$	k	$F(k)$	k	$F(k)$	k	$F(k)$
				$\sigma=0.1$					
0.00	−0.395	0.75	−0.107	1.50	0.401	2.25	0.805	3.00	1.097
0.05	−0.394	0.80	−0.074	1.55	0.433	2.30	0.828	3.05	1.114
0.10	−0.389	0.85	−0.040	1.60	0.463	2.35	0.850	3.10	1.130
0.15	−0.382	0.90	−0.008	1.65	0.493	2.40	0.871	3.15	1.146
0.20	−0.371	0.95	0.029	1.70	0.523	2.45	0.892	3.20	1.162
0.25	−0.358	1.00	0.064	1.75	0.551	2.50	0.913	3.25	1.178
0.30	−0.342	1.05	0.098	1.80	0.579	2.55	0.933	3.30	1.193
0.35	−0.324	1.10	0.133	1.85	0.607	2.60	0.952	3.35	1.208
0.40	−0.303	1.15	0.168	1.90	0.634	2.65	0.972	3.40	1.223
0.45	−0.280	1.20	0.203	1.95	0.660	2.70	0.991	3.45	1.238
0.50	−0.255	1.25	0.237	2.00	0.686	2.75	1.009	3.50	1.252
0.55	−0.229	1.30	0.271	2.05	0.711	2.80	1.028		From
0.60	−0.200	1.35	0.304	2.10	0.735	2.85	1.045		here
0.65	−0.170	1.40	0.337	2.15	0.759	2.90	1.063		on as
0.70	−0.139	1.45	0.370	2.20	0.782	2.95	1.080		in k
				$\sigma=0.2$					
0.00	−0.441	0.65	−0.201	1.30	0.262	1.95	0.660	2.60	0.954
0.05	−0.440	0.70	−0.168	1.35	0.296	2.00	0.686	2.65	0.973
0.10	−0.435	0.75	−0.134	1.40	0.330	2.05	0.711	2.70	0.992
0.15	−0.427	0.80	−0.099	1.45	0.364	2.10	0.736	2.75	1.010
0.20	−0.416	0.85	−0.063	1.50	0.396	2.15	0.760	2.80	1.028
0.25	−0.402	0.90	−0.027	1.55	0.428	2.20	0.783	2.85	1.046
0.30	−0.384	0.95	0.009	1.60	0.460	2.25	0.806	2.90	1.064
0.35	−0.365	1.00	0.046	1.65	0.490	2.30	0.829	2.95	1.081
0.40	−0.343	1.05	0.082	1.70	0.520	2.35	0.851	3.00	1.098
0.45	−0.318	1.10	0.119	1.75	0.550	2.40	0.872		From
0.50	−0.292	1.15	0.155	1.80	0.578	2.45	0.893		here
0.55	−0.263	1.20	0.191	1.85	0.606	2.50	0.914		on as
0.60	−0.233	1.25	0.227	1.90	0.633	2.55	0.934		in k

$\sigma = 0.3$

0.00	—0.527	1.00	0.005	2.00	0.678	3.00	1.096	4.00	1.385
0.05	—0.525	1.05	0.045	2.05	0.704	3.05	1.112	4.05	1.398
0.10	—0.520	1.10	0.084	2.10	0.729	3.10	1.129	4.10	1.410
0.15	—0.511	1.15	0.123	2.15	0.754	3.15	1.145	4.15	1.422
0.20	—0.498	1.20	0.161	2.20	0.778	3.20	1.161	4.20	1.434
0.25	—0.483	1.25	0.199	2.25	0.801	3.25	1.177	4.25	1.446
0.30	—0.464	1.30	0.236	2.30	0.824	3.30	1.192	4.30	1.458
0.35	—0.442	1.35	0.273	2.35	0.846	3.35	1.207	4.35	1.469
0.40	—0.418	1.40	0.308	2.40	0.868	3.40	1.222	4.40	1.481
0.45	—0.391	1.45	0.344	2.45	0.889	3.45	1.237	4.45	1.492
0.50	—0.362	1.50	0.378	2.50	0.910	3.50	1.251	4.50	1.503
0.55	—0.330	1.55	0.412	2.55	0.930	3.55	1.265	4.55	1.514
0.60	—0.297	1.60	0.444	2.60	0.950	3.60	1.280	4.60	1.526
0.65	—0.263	1.65	0.476	2.65	0.970	3.65	1.293	4.65	1.536
0.70	—0.226	1.70	0.507	2.70	0.989	3.70	1.307	4.70	1.547
0.75	—0.189	1.75	0.538	2.75	1.007	3.75	1.320	4.75	1.558
0.80	—0.151	1.80	0.567	2.80	1.026	3.80	1.334		From
0.85	—0.113	1.85	0.596	2.85	1.044	3.85	1.347		here
0.90	—0.074	1.90	0.624	2.90	1.061	3.90	1.360		on as
0.95	—0.034	1.95	0.651	2.95	1.079	3.95	1.373		ln *k*

$\sigma = 0.4$

0.00	—0.683	0.45	—0.527	0.90	—0.166	2.35	0.828	4.80	1.566
0.05	—0.681	0.50	—0.493	0.95	—0.121	2.40	0.851	4.85	1.576
0.10	—0.675	0.55	—0.457	1.00	—0.077	2.45	0.873	4.90	1.586
0.15	—0.664	0.60	—0.419	1.05	—0.033	3.50	1.244	4.95	1.597
0.20	—0.650	0.65	—0.380	1.10	0.010	3.55	1.259	6.00	1.791
0.25	—0.632	0.70	—0.339	1.15	0.053	3.60	1.273	6.05	1.799
0.30	—0.611	0.75	—0.296	1.20	0.096	3.65	1.287	6.10	1.807
0.35	—0.586	0.80	—0.253	2.25	0.781	3.70	1.301	6.15	1.816
0.40	—0.558	0.85	—0.210	2.30	0.805	4.75	1.555	6.20	1.824

k	*F* (*k*)	*k*	*F* (*k*)	*k*	*F* (*k*)	*k*	*F* (*k*)	*k*	*F* (*k*)
1.25	0.138	2.50	0.894	3.75	1.315	5.00	1.607	6.25	1.832
1.30	0.178	2.55	0.915	3.80	1.328	5.05	1.617	6.30	1.840
1.35	0.219	2.60	0.936	3.85	1.342	5.10	1.627	6.35	1.848
1.40	0.258	2.65	0.956	3.90	1.355	5.15	1.637	6.40	1.856
1.45	0.296	2.70	0.975	3.95	1.368	5.20	1.646	6.45	1.863
1.50	0.333	2.75	0.995	4.00	1.380	5.25	1.656	6.50	1.871
1.55	0.369	2.80	1.014	4.05	1.393	5.30	1.666	6.55	1.879
1.60	0.405	2.85	1.032	4.10	1.406	5.35	1.675	6.60	1.886
1.65	0.439	2.90	1.050	4.15	1.418	5.40	1.685	6.65	1.894
1.70	0.472	2.95	1.068	4.20	1.430	5.45	1.694	6.70	1.902
1.75	0.505	3.00	1.085	4.25	1.442	5.50	1.703	6.75	1.909
1.80	0.536	3.05	1.102	4.30	1.454	5.55	1.712	6.80	1.916
1.85	0.566	3.10	1.119	4.35	1.466	5.60	1.721	6.85	1.924
1.90	0.596	3.15	1.136	4.40	1.477	5.65	1.730	6.90	1.931
1.95	0.625	3.20	1.152	4.45	1.489	5.70	1.739	6.95	1.938
2.00	0.653	3.25	1.168	4.50	1.500	5.75	1.748	7.00	1.945
2.05	0.680	3.30	1.184	4.55	1.511	5.80	1.756		From
2.10	0.706	3.35	1.199	4.60	1.522	5.85	1.765		here
2.15	0.732	3.40	1.214	4.65	1.533	5.90	1.774		on as
2.20	0.757	3.45	1.229	4.70	1.544	5.95	1.782		ln *k*

difference, however, does not exceed 10% when $0 < \sigma \leqslant 0.2$.

In conclusion, we note that with slight modification all the methods given here can be applied to the solution of contact problems for elastic layers (not necessarily axisymmetric).

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